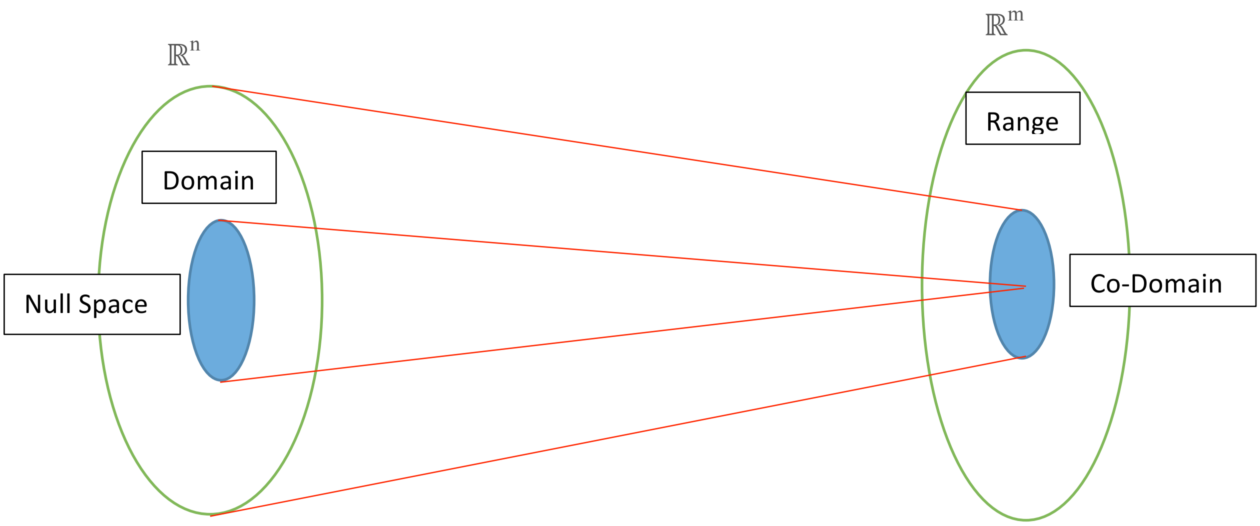
Computing for Data Sciences

Lecture 3

The below diagram represents a linear transformation of a vector into another vector by applying a Matrix on the vector.



$T_m: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (Linear Transformation)

A Matrix operation on a Vector of length, say n transforms it into a Vector of length, say m.

Domain of this transformation is a Vector space of dimension 'n'. Range is another Vector space of dimension 'm'.

$[M_{mxn}] * [v_{nx1}] = [W_{mx1}]$

Null Space of a Matrix M is defined as N(M) = {v: Mv = 0}

- If M is not a 1:1 map then Null Space of M will contain some vector $x \neq 0$.
- Zero vector will always be in the null space.

Column Space of A: The column space of a matrix A is the vector space made up of all linear combinations of the columns of A.

Span of Vectors: If there are k vectors $\{V_1, V_2, \ldots\ldots.. V_k\}$ then we define Span of the vectors as linear combinations of vectors.

$$\text{Span } \{ V_1, V_2, V_3\ldots..V_k \} = a_1V_1 + a_2V_2 + \ldots..+ a_kV_k \ : a_i \in \mathbb{R} \text{ and } V_i \in \mathbb{R}^n$$

Size of vector space depends on dimension of that space.

Span of $\{V_1\}$ => $\mathbb{R}_1$
Span of $\{V_1,V_2\}$ is either $\mathbb{R}_1$ or $\mathbb{R}_2$ depending on directions of V_1 and V_2.

In general if,

Span $\{v_1, v_2, \ldots\ldots, v_k\} = \mathbb{R}_k$ and Span $\{v_1, v_2, \ldots\ldots, v_k, v_{k+1}\} = \mathbb{R}_k$ then V_{k+1} is linearly dependent on $\{v_1, v_2, \ldots\ldots\ldots v_k\}$. which means

$V_{k+1} \in$ Span $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots, v_k\}$

Row reduction: Row reduction is done to check linear dependency in the matrix. We do this when trying to solve set of equations.

Process for row reduction:

The principle involved in row reduction of matrices are equivalent to those we used in the elimination method of solving systems of equations. That is, we are allowed to

1. Multiply a row by a non-zero constant.
2. Add one row to another.
3. Interchange between rows
4. Add a multiple of one row to another.

Exampl Reduce $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & 4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ Your goal is to transform it to $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$

$$\underset{R_1 + R_2 \Rightarrow R_2}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & 4 \\ 2 & -5 & 5 & 17 \end{bmatrix}} \rightarrow \underset{-2R_1 + R_3 \Rightarrow R_3}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 13 \\ 2 & -5 & 5 & 17 \end{bmatrix}} \rightarrow \underset{2R_2 + R_1 \Rightarrow R_1}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 13 \\ 0 & -1 & -1 & -1 \end{bmatrix}} \rightarrow \underset{R_2 + R_3 \Rightarrow R_3}{\begin{bmatrix} 1 & 0 & 9 & 35 \\ 0 & 1 & 3 & 13 \\ 0 & -1 & -1 & -1 \end{bmatrix}}$$

$$\underset{\frac{1}{2} R_3 \Rightarrow R_3}{\begin{bmatrix} 1 & 0 & 9 & 35 \\ 0 & 1 & 3 & 13 \\ 0 & 0 & 2 & 12 \end{bmatrix}} \rightarrow \underset{-3R_3 + R_2 \Rightarrow R_2}{\begin{bmatrix} 1 & 0 & 9 & 35 \\ 0 & 1 & 3 & 13 \\ 0 & 0 & 1 & 6 \end{bmatrix}} \rightarrow \underset{-9R_3 + R_1 \Rightarrow R_1}{\begin{bmatrix} 1 & 0 & 9 & 35 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 6 \end{bmatrix}} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

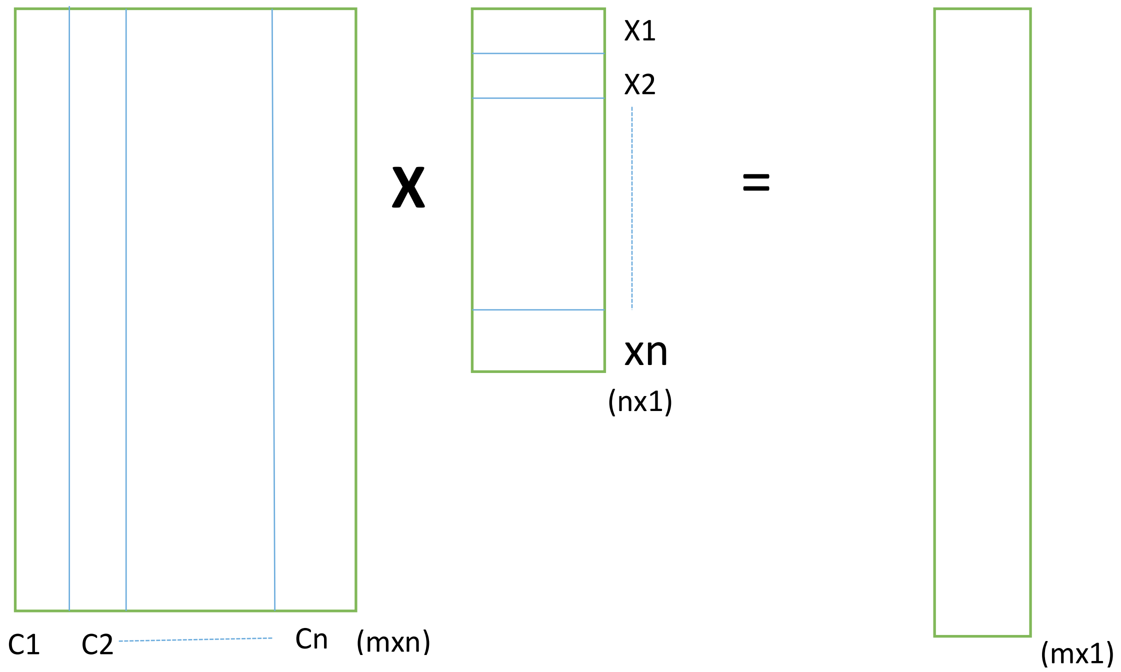
Image Source: https://www.ccis.edu/departments/mathcenter/matrix.pdf

Given equation AX = B is solvable for a particular value of 'B' if B lies in the column space of A. For any value of B, question comes whether A is an invertible matrix. If A is onto map then solution exists.

$T_m: \mathbb{R}_n \rightarrow \mathbb{R}^m$

- If n<m then this transformation can't be onto. In this case we have more equations than unknowns.
- If m=n then all n vectors have to be linearly independent to find the solution.
- If n>m then this transformation may be onto and might have infinite solutions. If some vectors here are linearly dependant, then they can be removed so that 'n' decreases.

We can visualize Matrix vector multiplication as below. Below picture shows a matrix A of size (m x n) being multiplied by a vector x or size (n x 1). C_1, C_2,……Cn are columns of matrix A and x_1,x_2,……xn are rows of vector x. Here x_1, x_2,….xn are not vectors but they are components of vector x.



Here right hand side is of the form

$= x_1*C_1 + x_2*C_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + x_n*C_n$

So the right side is a liner combination of the columns of matrix A. The right hand size can't have size more than n, Because addition of one vector can add maximum one dimension.

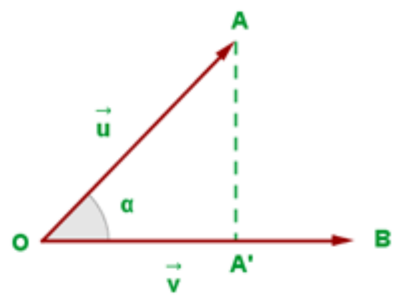
<u>Rank of a Matrix from its row-reduced **echelon** form</u>

On row-reduced echelon form the first non-zero entry in each row is called the **pivot**. The number of pivotal entries defines the number of non-zero rows (Good rows); hence number of pivotal entries is a measure of the row-space of the original matrix. As these rows cannot be turned to zero by taking any linear combinations, these rows of the original matrix are linearly independent. On the other hand, any column containing a pivotal entry must be linearly independent from all the previous columns as no other previous columns contain any non-zero entry on that row. Hence the number of pivotal entry also defines the column rank of a matrix.
So for any matrix

Row Rank = Column rank = Number of pivotal entries in Echelon form

Projection:

Scalar Projection



$$\cos\alpha = \frac{OA'}{|\vec{u}|} \qquad OA' = |\vec{u}| \cdot \cos\alpha$$

$$\vec{u} \cdot \vec{v} = |\vec{v}| \cdot OA' \qquad OA' = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

Vector Projection

The vector projection is the unit vector of $\vec{v}$ by the scalar projection of u on v.

$$proj_v u = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} \qquad proj_v u = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

The scalar projection of u on v is the <u>magnitude</u> of the vector projection of u on v.

Source: http://www.vitutor.com/geometry/vec/vector_projection.html

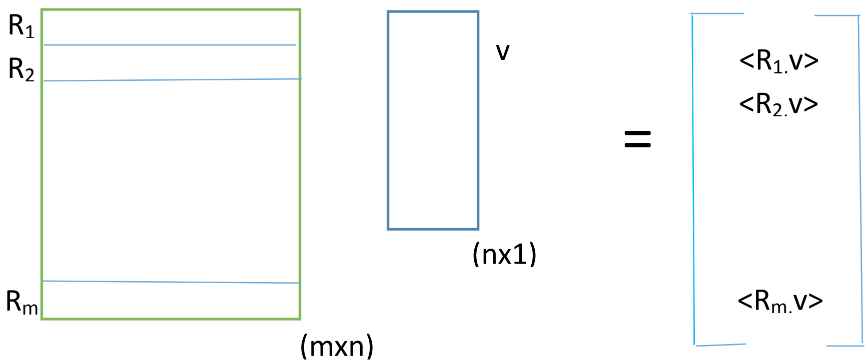
The dot product of two vectors $\overrightarrow{V_1}, \overrightarrow{V_2}$ can also be represented by inner-product of the two vectors

$$\overrightarrow{\boldsymbol{V_1}}.\overrightarrow{\boldsymbol{V_2}} = \langle\overrightarrow{\boldsymbol{V_1}}, \overrightarrow{\boldsymbol{V_2}}\rangle$$

- If V_1 and V_2 are in same direction then projection of V_1 on V_2 will be of length $||V1||$
- If V_1 and V_2 are orthogonal then projection of V_1 on V_2 will be of length 0.

$$\boldsymbol{V_1} \perp \boldsymbol{V_2} \Leftrightarrow \langle \boldsymbol{V_1}, \boldsymbol{V_2}\rangle = \boldsymbol{0}$$

If there is a matrix A of dimension (m x n), which we multiply to vector v of dimension (n x 1). Multiplication can be represented as below,



$R_1, R_2, \ldots, R_m$ are rows of matrix A. On the right side on each row we get inner product of corresponding row with the vector.

If vector V is in null space of matrix A, then each of the row of right side vector will be zero. It implies from our previous discussion that vector v is orthogonal to each of the row vectors of the matrix.

- ⇨ $V \in N(A) \perp R_i$ for $1 < i < m$ where R_i is i^{th} row of matrix A.
- ⇨ Null space is orthogonal to row space for any matrix.

$\Longrightarrow N(A) \perp R(A)$

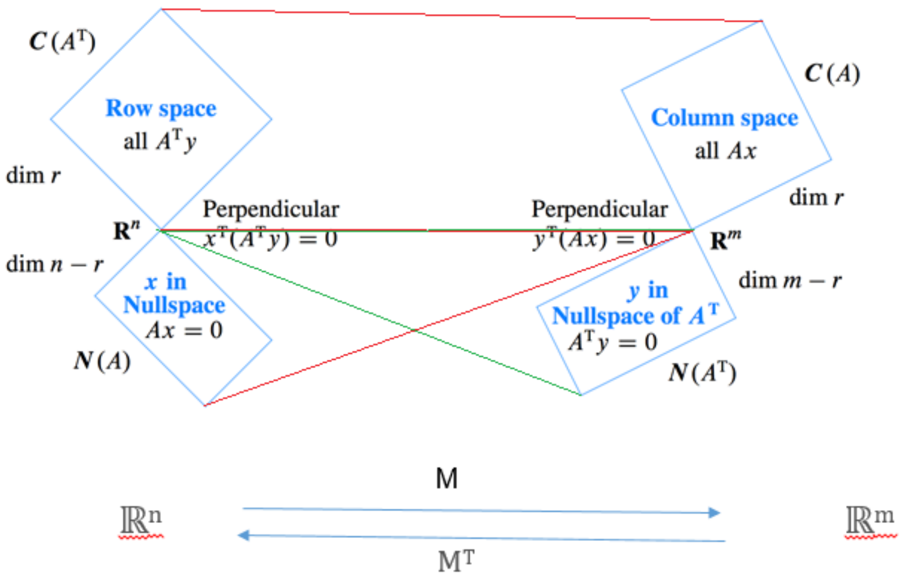
Now Column space of matrix A can be described as Row space of A^T.and vice versa.

$\Longrightarrow N(A) \perp C(A^T)$

In similar fashion, Row space of A^T will be orthogonal to Null space of A^T

$\Longrightarrow C(A) \perp N(A^T)$

These relationships can be described by the following pictorial representation:



The practical application is to take a huge matrix, throw out the Null Space, and model it such that a small matrix does the same thing as the bigger matrix with some acceptable degree of error.