Assignment 2

Posted on 12 Feb 2016 | Clarify doubts by 16 Feb 2016 | Submit by 19 Feb 2016

This is a *Group Assignment* – each group (two students) should submit a single set of solutions.

The solutions may be submitted either as a clearly legible hand-written document, or as a single LaTeX generated PDF document. In case of PDF submission, the filename should be `assign2_groupXX.pdf`, where `XX` is the serial number of the group. Be cogent, but concise.

Attempt all problems. This assignment is worth 50 points in total.

Background: If $X \in \mathbb{R}^{n\times n}$ is nonsingular, then the map $A \mapsto X^{-1}AX$ for $A \in \mathbb{R}^{n\times n}$ is called a *similarity transformation* of A. Two matrices A and B are *similar* if there is a similarity transformation relating one to the other, i.e., if there exists a nonsingular $X \in \mathbb{R}^{n\times n}$ such that $B = X^{-1}AX$. If $Q \in \mathbb{R}^{n\times n}$ is orthogonal, i.e., if $QQ^T = Q^TQ = I$, then the map $A \mapsto Q^TAQ$ for $A \in \mathbb{R}^{n\times n}$ is called a *unitary similarity transformation* of A.

A square matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonalizable* if it is similar to a diagonal matrix, i.e., if there exists a nonsingular $X \in \mathbb{R}^{n\times n}$ such that $D = X^{-1}AX$ is a diagonal matrix. A square matrix $A \in \mathbb{R}^{n\times n}$ is called *unitarily diagonalizable* if there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ such that $D = Q^TAQ$ is a diagonal matrix.

A related decomposition of $A \in \mathbb{R}^{n\times n}$ is the *Schur factorization* $A = QTQ^T$, where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, and $T \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix. $A \in \mathbb{R}^{n\times n}$ is called *Schur factorizable* if there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ such that $T = Q^TAQ$ is an upper-triangular matrix.

Problem 1

[10 + 10 + 10 + 20 = 50]

A. Prove that two *similar* matrices have identical eigenvalues. As a corollary, prove that the eigenvalues of a *diagonalizable* matrix A is revealed by its diagonal reduction $D = X^{-1}AX$. Does the *Schur factorization* of a square matrix $A \in \mathbb{R}^{n\times n}$ reveal its eigenvalues? Justify.

B. Is every square matrix $A \in \mathbb{R}^{n\times n}$ unitarily diagonalizable? Provide a necessary and sufficient condition for a square matrix $A \in \mathbb{R}^{n\times n}$ to be unitarily diagonalizable, with justification.

C. Is every square matrix $A \in \mathbb{R}^{n\times n}$ Schur factorizable? Provide a necessary and sufficient condition for a square matrix $A \in \mathbb{R}^{n\times n}$ to be Schur factorizable, with justification.

D. Is it possible to diagonalize or Schur factorize a square matrix $A \in \mathbb{R}^{n\times n}$ through a finite number of *unitary similarity transformation* on A? If yes, provide the description of such an algorithm, with justification. If not, provide a justification why it may not be possible.