INDIAN STATISTICAL INSTITUTE

Mid-Semester of Second Semester Examination : 2015–16

Course : Bachelor of Statistics (Hons.)

Subject : Numerical Analysis : BStat-I

Date : 22 February 2016 | Maximum Marks : 60 | Duration : 3 Hours

Attempt any two from the first three problems. In addition, attempt the bonus problem.

Problem 1 **[25]**

Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, the QR decomposition of the matrix satisfies $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{R}^{m \times n}$ is an orthogonal matrix, and $\mathbf{R} \in \mathbb{R}^{n \times n}$ is an upper-triangular matrix.

A. Describe an algorithm that orthogonalizes $\mathbf{A}$ through successive right multiplication by upper triangular matrices, such that $\mathbf{A}\mathbf{R}_1\mathbf{R}_2 \cdots \mathbf{R}_n = \mathbf{Q}$, and $\mathbf{R} = (\mathbf{R}_1\mathbf{R}_2 \cdots \mathbf{R}_n)^{-1}$. [10]

B. Describe an algorithm that triangularizes $\mathbf{A}$ through successive left multiplication by orthogonal matrices, such that $\mathbf{Q}_n \cdots \mathbf{Q}_2\mathbf{Q}_1\mathbf{A} = \mathbf{R}$, and $\mathbf{Q} = (\mathbf{Q}_n \cdots \mathbf{Q}_2\mathbf{Q}_1)^{-1}$. [10]

C. Which of the above two algorithms will you prefer for the QR decomposition of an arbitrary $m \times n$ real matrix $\mathbf{A}$? Justify you answer. [5]

Problem 2 **[25]**

A. Prove that all eigenvalues of a real symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are real. [4]

B. Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a real symmetric matrix, having n distinct eigenvalues, which satisfy the ordering $|\lambda_1| > |\lambda_2| > \cdots |\lambda_n| > 0$. Describe an *iterative* algorithm to compute the largest (in absolute value) eigenvalue λ_1 of $\mathbf{A}$. [7]

C. Propose a similar iterative method, by introducing a modification in the above algorithm, to compute the smallest (in absolute value) eigenvalue λ_n of $\mathbf{A}$. [7]

D. Given a real number σ, propose a similar iterative method, with a modification in the above algorithm, to compute the eigenvalue λ_k of $\mathbf{A}$ that is closest (in absolute value) to σ. [7]

Note: In this problem, do not use an algorithm that tries to find *all* eigenvalues of $\mathbf{A}$ simultaneously.

Problem 3 **[25]**

A. Suppose that the *full* Singular Value Decomposition of $\mathbf{A} \in \mathbb{R}^{m \times n}$ results in:

$$\mathbf{A} = \begin{bmatrix} | & & | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_r & \cdots & \mathbf{u}_m \\ | & & | & & | \end{bmatrix} \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} | & & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_r & \cdots & \mathbf{v}_n \\ | & & | & & | \end{bmatrix}^T$$

What are the dimensions of each matrix on the right-hand-side of this decomposition? Discuss the connection of these matrices with the RowSpace, ColSpace and NullSpace of $\mathbf{A}$. How can you determine the Rank of $\mathbf{A}$ given this SVD representation? [3 + 6 + 1]

B. As per the SVD of $\mathbf{A}$, determine the dimension and rank of each of the matrices $\mathbf{Z}_i = \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, where $1 \le i \le r$. Is it possible to reconstruct $\mathbf{A}$ given the matrices $\mathbf{Z}_i$ for $1 \le i \le k$, where k is strictly less than r? If so, provide such a construction. If not, provide an *approximate* reconstruction $\mathbf{A_k}$ of $\mathbf{A}$ using the available matrices $\mathbf{Z}_i$ for $1 \le i \le k$. Provide an expression for $||\mathbf{A} - \mathbf{A_k}||_2 = \max_{||\mathbf{x}||_2 = 1} ||(\mathbf{A} - \mathbf{A_k})\mathbf{x}||_2$ in terms of the singular values σ_i. [2 + 3 + 5]

C. The *pseudo-inverse* of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is generally expressed as $\mathbf{A}^+ = (\mathbf{A^T A})^{-1} \mathbf{A}^T$. Provide a computationally simple construction of $\mathbf{A}^+$ from the singular value decomposition of $\mathbf{A}$. [5]

Bonus Problem **[10]**

Represent a book in the form of an $m \times n$ real matrix $\mathbf{B}$, where m is the total number of sentences in the book and n is the total number of distinct words in the book, such that the entry $\mathbf{B}[i, j]$ in this matrix represents the frequency of occurrence of the j-th word W_j in the i-th sentence S_i.

Importance of the words and sentences are denoted by *scores*. Score u_i of sentence S_i is proportional to the weighted sum of scores of the words in it. Score v_j of word W_j is proportional to the weighted sum of scores of the sentences it is contained in. This may be expressed as follows.

$$u_i \propto \sum_{j=1}^{n} \mathbf{B}[i, j] \cdot v_j \quad \text{for } i = 1, 2, \ldots, m \qquad\qquad v_j \propto \sum_{i=1}^{m} \mathbf{B}[i, j] \cdot u_i \quad \text{for } j = 1, 2, \ldots, n$$

Devise an efficient strategy to identify 10 *keywords* (i.e., the most important words) from the book.

Good luck! ☺